# Machine Learning II

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**Outline**

- Literature
- Notation
- Pseudo-Boolean functions
- Multi-linear polynomial forms
  - Existence and uniqueness
  - Reduction of PBO to QPBO
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  - Bounds
  - Weak persistency
  - Complementation and the Floor Dual Bound

This lecture is based on the publications

- E. Boros, P. L. Hammer, X. Sun: Network flows and minimization of quadratic pseudo-Boolean functions. RUTCOR Research Report 17-1991
- E. Boros, P. L. Hammer: Pseudo-Boolean optimization. Discrete Applied Mathematics 123(1–3): 155–225 (2002)
- E. Boros, P. L. Hammer, R. Sun, G. Tavares: A max-flow approach to improved lower bounds for quadratic unconstrained binary optimization (QUBO). Discrete Optimization 5(2): 501–529 (2008)

## Definition 1

*For any $n \in \mathbb{N}$, any $d \in \{0, \dots, n\}$, let*

$$J_{nd} := \bigcup_{m=0}^{d} \binom{\{1, \dots, n\}}{d} \qquad C_{nd} := \mathbb{R}^{J_{nd}} \tag{1}$$

*and call any $c \in C_{nd}$ an $n$-variate* **multi-linear polynomial form** *of degree at most $d$.*

**Example.** For $n = d = 2$, we have

$$\begin{aligned} J_{22} &= \bigcup_{m=0}^{2} \binom{\{1,2\}}{m} \\ &= \binom{\{1,2\}}{0} \cup \binom{\{1,2\}}{1} \cup \binom{\{1,2\}}{2} \\ &= \{\emptyset\} \cup \{\{1\},\{2\}\} \cup \{\{1,2\}\} \\ &= \{\emptyset, \{1\}, \{2\}, \{1,2\}\} \end{aligned}$$

## Definition 2

*For any* $n \in \mathbb{N}$, *any* $d \in \{0, \ldots, n\}$ *and any* $c \in C_{nd}$, $f_c : \{0,1\}^n \to \mathbb{R}$ *such that*

$$\forall x \in \{0,1\}^n : \quad f_c(x) := \sum_{m=0}^{d} \sum_{J \in \binom{\{1,\ldots,n\}}{m}} c_J \prod_{j \in J} x_j \tag{2}$$

*is called the* **PBF defined by** $c$.

**Example.** For any $c \in C_{22}$, $f_c : \{0,1\}^2 \to \mathbb{R}$ is such that

$$\forall x \in \{0,1\}^2 : \quad f_c(x_1, x_2) = c_\emptyset + c_{\{1\}} x_1 + c_{\{2\}} x_2 + c_{\{1,2\}} x_1 x_2 \ .$$

## Lemma 1

*Every PBF has a unique multi-linear polynomial form. More precisely,*

$$\forall n \in \mathbb{N} \quad \forall f : \{0,1\}^n \to \mathbb{R} \quad \exists_1 c \in C_{nn} \quad f = f_c \ . \tag{3}$$

**Example.** For $n = d = 2$ and any $f : \{0,1\}^2 \to \mathbb{R}$, the existence of a $c \in C_{22}$ such that $f = f_c$ means

$$\forall x \in \{0,1\}^2 \quad f(x_1, x_2) = c_\emptyset + c_{\{1\}} x_1 + c_{\{2\}} x_2 + c_{\{1,2\}} x_1 x_2 \ .$$

Explicitly,

$$\begin{aligned} f(0,0) &= c_\emptyset \\ f(1,0) &= c_\emptyset + c_{\{1\}} \\ f(0,1) &= c_\emptyset \qquad\quad + c_{\{2\}} \\ f(1,1) &= c_\emptyset + c_{\{1\}} + c_{\{2\}} + c_{\{1,2\}} \ . \end{aligned}$$

In this example, a suitable $c$ exists and is defined uniquely by $f$.

**Proof.**

- For any $J \subseteq \{1, \ldots, n\}$, let $x^J \in \{0,1\}^n$ such that

$$\forall j \in \{1, \ldots, n\}\colon \quad x_j^J = \begin{cases} 1 & \text{if } j \in J \\ 0 & \text{otherwise} \end{cases} \quad .$$

- Now,

$$\forall x \in \{0,1\}^n\colon \qquad f(x) = \sum_{J \subseteq \{1,\ldots,n\}} c_J \prod_{j \in J} x_j$$

  is written equivalently as

$$\begin{aligned} & f(x^{\emptyset}) = c_{\emptyset} \\ \forall J \neq \emptyset\colon \quad & f(x^J) = c_J + \sum_{J' \subset J} c_{J'} \quad . \end{aligned}$$

- Thus, $c$ is defined uniquely (by induction over the cardinality of $J$).

Definition 3

*For any* $n \in \mathbb{N}$ *and any* $d \in \{0, \ldots, n\}$, *let*

$$F_{nd} := \{f : \{0,1\}^n \to \mathbb{R} \mid \exists c \in C_{nd} : \; f = f_c\} \tag{4}$$

*and call any* $f \in F_{nd}$ *an* $n$-**variate PBF of degree at most** $d$.

*In addition, call any* $f \in F_{n2}$ *a* **quadratic PBF (QPBF)**.

**Note.** For any $n \in \mathbb{N}$, $F_{nn}$ is the set of all $n$-variate PBFs (by Lemma 1).

- **Pseudo-Boolean Optimization (PBO)**: Given $n \in \mathbb{N}$ and $f : \{0,1\}^n \to \mathbb{R}$,

$$\min_{x \in \{0,1\}^n} f(x) \ . \tag{5}$$

- **Quadratic Pseudo-Boolean Optimization (QPBO)**: Given $n \in \mathbb{N}$ and $f \in F_{n2}$,

$$\min_{x \in \{0,1\}^n} f(x) \ . \tag{6}$$

- Is QPBO less complex than PBO?

### Definition 4

*For any $n \in \mathbb{N}$ and any $c \in C_{nn}$, define the* **size** *of $c$ as*

$$\mathrm{size}(c) := \sum_{J \subseteq \{1,\ldots,n\}\colon\, c_J \neq 0} |J| \ . \qquad (7)$$

Lemma 2

*For any* $x, y, z \in \{0, 1\}$*:*

$$z = xy \quad \Leftrightarrow \quad xy - 2xz - 2yz + 3z = 0 \ , \tag{8}$$

$$z \neq xy \quad \Leftrightarrow \quad xy - 2xz - 2yz + 3z > 0 \ . \tag{9}$$

**Proof.** By verifying equivalence for all eight cases.

## Algorithm 1 (Boros and Hammer 2001)

**Input**: $c \in C_{nn}$
**Output**: $c' \in C_{n2}$
$M := 1 + 2\sum_{J\subseteq\{1,\ldots,n\}} |c_J|$
$m := n$
$c^m := c$
**while** *there exists a* $J \subseteq \{1,\ldots,n\}$ *such that* $|J| > 2$ *and* $c^m_J \neq 0$
  *Choose* $j, k \in J$ *such that* $j \neq k$
  $c^{m+1} := c^m$
  $c^{m+1}_{\{j,k\}} := c^{m+1}_{\{j,k\}} + M$
  $c^{m+1}_{\{j,m+1\}} := -2M$
  $c^{m+1}_{\{k,m+1\}} := -2M$
  $c^{m+1}_{\{m+1\}} := 3M$
  **for all** $\{j,k\} \subseteq J' \subseteq \{1,\ldots,n\}$ *such that* $c^{m+1}_{J'} \neq 0$
    $c^{m+1}_{J'-\{j,k\}\cup\{m+1\}} := c^{m+1}_{J'}$
    $c^{m+1}_{J'} := 0$
  $m := m + 1$
$c' := c^m$

## Theorem 1

- *Algorithm 1 terminates in polynomial time in* $\mathrm{size}(c)$.
- $\mathrm{size}(c')$ *is polynomially bounded by* $\mathrm{size}(c)$.
- *The multi-linear quadratic form* $c'$ *is such that* $\forall \hat{x} \in \mathbb{R}^n$:

$$\begin{aligned} & \hat{x} \in \operatorname*{argmin}_{x \in \{0,1\}^n} f_c(x) \\ \Rightarrow \quad & \exists \hat{x}' \in \{0,1\}^m \left( \hat{x}'_{\{1,\dots,n\}} = \hat{x}_{\{1,\dots,n\}} \wedge \hat{x}' \in \operatorname*{argmin}_{x' \in \{0,1\}^m} f_{c'}(x') \right) \quad . \end{aligned} \tag{10}$$

**Proof.**

- The algorithm replaces the occurrence of $x_j x_k$ by $x_{m+1}$ and adds the form $M(x_j x_k - 2x_j x_{m+1} - 2x_k x_{m+1} + 3x_{m+1})$.
  - If $x_{m+1} = x_j x_k$,
$$f^{m+1}(x_1, \ldots, x_{m+1}) = f^m(x_1, \ldots, x_n) \leq \max_{x' \in \{0,1\}^n} f^m(x') < M/2 \ .$$
  - If $x_{m+1} \neq x_j x_k$,
$$f^{m+1}(x_1, \ldots, x_{m+1}) \geq M/2$$

  (by Lemma 2 and by definition of $M$).
- For every iteration $m$,
$$|\{J \subseteq \{1, \ldots, n\} | |J| > 2 \wedge c_J^{m+1} \neq 0\}| < |\{J \subseteq \{1, \ldots, n\} | |J| > 2 \wedge c_J^m \neq 0\}|$$

  which proves the complexity claims.

Summary

- Every PBF has a unique multi-linear polynomial form.
- PBO is polynomially reducible to QPBO.

## Definition 5

*For any $n \in \mathbb{N}$ and any $d \in \{0, \ldots, n\}$, let*

$$K^+_{nd} := \{(K^1, K^0) | K^1, K^0 \subseteq \{1, \ldots, n\} \wedge K^1 \cap K^0 = \emptyset \wedge |K^1| + |K^0| = d\}$$

$$J^+_{nd} := \bigcup_{m=0}^{d} K^+_{nm}$$

$$C^+_{nd} := \{c : J^+_{nd} \to \mathbb{R} \mid \forall j \in J^+_{nd} \setminus \{(\emptyset, \emptyset)\} : \ 0 \leq c_j\}$$

*and call any $c \in C^+_{nd}$ an $n$-variate* **posiform** *of degree at most $d$.*

**Example.** For $n = d = 2$,

$$\begin{aligned} J^+_{22} = & \ \ \{ \ (\emptyset, \emptyset) \ \} \\ & \cup \{ \ (\{1\}, \emptyset), \ (\emptyset, \{1\}), \ (\{2\}, \emptyset), \ (\emptyset, \{2\}) \ \} \\ & \cup \{ \ (\{1,2\}, \emptyset), \ (\{1\}, \{2\}), \ (\{2\}, \{1\}), \ (\emptyset, \{1,2\}) \ \} \end{aligned}$$

## Definition 6

*For any $n \in \mathbb{N}$, any $d \in \{0, \dots, n\}$ and any $c \in C^+_{nd}$, $f_c : \{0,1\}^n \to \mathbb{R}$ such that*

$$\forall x \in \{0,1\}^n \qquad f_c(x) := \sum_{(J^1, J^0) \in J^+_{nd}} c_{J^1 J^0} \prod_{j \in J^1} x_j \prod_{j' \in J^0} (1 - x'_j) \tag{11}$$

*is called the* **PBF defined by** $c$.

**Example.** For any $c \in C^+_{22}$, $f_c : \{0,1\}^2 \to \mathbb{R}$ is such that $\forall x \in \{0,1\}^2$:

$$\begin{aligned} f(x) = \quad & c_{\emptyset\emptyset} \\ & + c_{\{1\}\emptyset} x_1 + c_{\emptyset\{1\}}(1 - x_1) + c_{\{2\}\emptyset} x_2 + c_{\emptyset\{2\}}(1 - x_2) \\ & + c_{\{1,2\}\emptyset} x_1 x_2 + c_{\{1\}\{2\}} x_1 (1 - x_2) + c_{\{2\}\{1\}}(1 - x_1) x_2 \\ & \quad + c_{\emptyset\{1,2\}}(1 - x_1)(1 - x_2) \ . \end{aligned}$$

## Definition 7

*For any* $n \in \mathbb{N}$ *and any* $f : \{0,1\}^n \to \mathbb{R}$*, the posiform defined by*

$$\forall x \in \{0,1\}^n: \quad K_x^1 := \{j \in \{1,\dots,n\} | x_j = 1\}$$
$$K_x^0 := \{j \in \{1,\dots,n\} | x_j = 0\}$$

*and*

$$J := \{(\emptyset,\emptyset)\} \cup \bigcup_{x \in \{0,1\}^n} \{(K_x^1, K_x^0)\}$$

*and* $c : J \to \mathbb{R}$ *such that*

$$c_{\emptyset\emptyset} := \min_{x \in \{0,1\}^n} f(x)$$
$$\forall x \in \{0,1\}^n \quad c_{K_x^1 K_x^0} := f(x) - c_{\emptyset\emptyset}$$

*is called* **min-term posiform** *of* $f$*.*

Lemma 3

*For any* $n \in \mathbb{N}$ *and any* $f : \{0,1\}^n \to \mathbb{R}$*, the min-term posiform* $c$ *of* $f$ *holds* $f_c = f$*.*

Corollary 1

*For any* $n \in \mathbb{N}$ *and any* $f : \{0,1\}^n \to \mathbb{R}$*, there exists a posiform* $c \in C^{+}_{nn}$ *such that* $f_c = f$*.*

**Proof of Lemma 3.**

- Let $n \in \mathbb{N}$ and $f : \{0,1\}^n \to \mathbb{R}$. Moreover, let $c : J \to \mathbb{R}$ the min-term posiform of $f$.
- $c$ is a posiform (by definition).
- Let $g : \{0,1\}^n \to \mathbb{R}$ be the PBF defined by this posiform.
- Then, for any $x \in \{0,1\}^n$,

  $$(J^1, J^0) \in \{(\emptyset, \emptyset), (K_x^1, K_x^0)\} \subseteq J$$

  are the only elements of $J$ for which

  $$0 \neq \prod_{j \in J^1} x_j \prod_{j' \in J^0} (1 - x'_j) = 1 \ .$$

- Thus,

$$\begin{aligned} \forall x \in \{0,1\}^n \qquad g(x) &= c_{\emptyset\emptyset} + c_{K_x^1 K_x^0} \\ &= c_{\emptyset\emptyset} + f(x) - c_{\emptyset\emptyset} \qquad \text{(by definition of } c\text{)} \\ &= f(x) \ . \end{aligned}$$

**Note.** Unlike multi-linear polynomial forms, posiforms of PBFs need not be unique, e.g., $x_1 = x_1 x_2 + x_1(1 - x_2)$.

## Definition 8

*For any* $n \in \mathbb{N}$*, any* $f : \{0,1\}^n \to \mathbb{R}$ *and any* $d \in \{0, \dots, n\}$*, let*

$$C^+_{nd}(f) := \left\{ c \in C^+_{nd} \mid f_c = f \right\} \quad . \tag{12}$$

**Note.** For any $n \in \mathbb{N}$ and any $f : \{0,1\}^n \to \mathbb{R}$, $C^+_{nn}(f)$ contains at least the min-term posiform of $f$.

Lemma 4

$$\forall n \in \mathbb{N} \quad \forall f : \{0,1\}^n \to \mathbb{R} \quad \forall c \in C^+_{nn}(f) \quad \forall x \in \{0,1\}^n \quad c_{\emptyset\emptyset} \leq f(x) \ .$$

**Proof.**

- By definition, we have, for all $x \in \{0,1\}^n$,

$$f(x) = \sum_{m=0}^{d} \sum_{(K^1,K^0)\in K^+_{nm}} c_{K^1K^0} \prod_{j\in K^1} x_j \prod_{j'\in K^0} (1-x'_j)$$

$$= c_{\emptyset\emptyset} + \sum_{m=1}^{d} \sum_{(K^1,K^0)\in K^+_{nm}} c_{K^1K^0} \prod_{j\in K^1} x_j \prod_{j'\in K^0} (1-x'_j) \ ,$$

  and all coefficients $c_{K^1K^0}$ in the second sum are non-negative.
- Therefore, the second sum is non-negative.
- Thus,

$$\forall x \in \{0,1\}^n \qquad f(x) \geq c_{\emptyset\emptyset} \ .$$

Definition 9

*For any posiform* $c : J \to \mathbb{R}$*, a pair* $(S, y)$ *such that* $S \subseteq \{1, \ldots, n\}$ *and* $y : S \to \{0, 1\}$ *is called a* **contractor** *of* $c$ *iff*

$$
\begin{aligned}
\forall (J^1, J^0) \in J \qquad & (J^1 \cap S = \emptyset \quad \wedge \quad J^0 \cap S = \emptyset) \\
& \vee (\exists j \in J^1 \cap S \quad y_j = 0) \\
& \vee (\exists j \in J^0 \cap S \quad y_j = 1) \ . \qquad (13)
\end{aligned}
$$

## Lemma 5

*For any* $n \in \mathbb{N}$*, any* $f : \{0,1\}^n \to \mathbb{R}$*, any posiform* $c \in C^+_{nn}(f)$*, any contractor* $(S, y)$ *of* $c$ *and* $t_{S,y} : \{0,1\}^n \to \{0,1\}^n$ *such that*

$$\forall x \in \{0,1\}^n \quad \forall j \in \{1,\dots,n\} \quad (t_{S,y}(x))_j = \begin{cases} y_j & \text{if } j \in S \\ x_j & \text{otherwise} \end{cases} \tag{14}$$

*holds*

$$\forall x \in \{0,1\}^n \quad f(t_{S,y}(x)) \leq f(x) \ . \tag{15}$$

## Corollary 2 (weak persistency)

$$\hat{x} \in \operatorname*{argmin}_{x \in \{0,1\}^n} f(x) \quad \Rightarrow \quad t_{S,y}(\hat{x}) \in \operatorname*{argmin}_{x \in \{0,1\}^n} f(x) \tag{16}$$

**Proof of Lemma 5.**

- Let $J^{\bar{S}} := \{(J^1, J^0) \in J^+_{nn} \mid J^1 \cap S = J^0 \cap S = \emptyset\}$ and $J^S := J - J^{\bar{S}}$.
- By definition,

$$\forall x \in \{0,1\}^n \qquad f(x) = \underbrace{\sum_{(J^1,J^0)\in J^S} c_{J^1 J^0} \prod_{j\in J^1} x_j \prod_{j'\in J^0} (1 - x'_j)}_{=: f^S(x)} + \underbrace{\sum_{(J^1,J^0)\in J^{\bar{S}}} c_{J^1 J^0} \prod_{j\in J^1} x_j \prod_{j'\in J^0} (1 - x'_j)}_{=: f^{\bar{S}}(x)} \ .$$

- Furthermore,

$$\forall x \in \{0,1\}^n \qquad f^S(t_{S,y}(x)) = 0 \qquad \text{(by definition)}$$

$$0 \leq f^S(x) \qquad \text{(because } (\emptyset, \emptyset) \notin J^S\text{)}$$

$$f^{\bar{S}}(t_{S,y}(x)) = f^{\bar{S}}(x) \qquad \text{(by definition)}$$

Summary

- Every PBF has a posiform
- The posiform of a PBF need not be unique
- For every PBF $f$ and every posiform $c$ of $f$
  - $c_{\emptyset\emptyset}$ is a lower bound on the minimum of $f$
  - weak persistency holds at any contractor of $c$

For any $n \in \mathbb{N}$, consider $n$-variate **quadratic** forms:

- any **multi-linear polynomial form** $c \in C_{n2}$ and $f_c : \{0,1\}^2 \to \mathbb{R}$, i.e., for all $x \in \{0,1\}^n$,

$$f_c(x) = c_\emptyset + \sum_{j \in \{1,\ldots,n\}} c_{\{j\}} x_j + \sum_{\{j,k\} \in \binom{\{1,\ldots,n\}}{2}} c_{\{j,k\}} x_j x_k$$

- any **posiform** $c' \in C^+_{n2}$ and $f'_c : \{0,1\}^2 \to \mathbb{R}$, i.e., for all $x \in \{0,1\}^n$,

$$\begin{aligned} f'_{c'}(x) = c'_{\emptyset\emptyset} &+ \sum_{j \in \{1,\ldots,n\}} \Big( c'_{\{j\}\emptyset} x_j + c'_{\emptyset\{j\}} (1 - x_j) \Big) \\ &+ \sum_{\{j,k\} \in \binom{\{1,\ldots,n\}}{2}} \big( c'_{\{j,k\}\emptyset} x_j x_k + c'_{\{j\}\{k\}} x_j (1 - x_k) \\ &\qquad + c'_{\{k\}\{j\}} x_k (1 - x_j) + c'_{\emptyset\{j,k\}} (1 - x_j)(1 - x_k) \big) \end{aligned}$$

## Lemma 6

*For any* $n \in \mathbb{N}$*, any QPBF* $f : \{0,1\}^n \to \mathbb{R}$*, the* $c \in C_{n2}$ *such that* $f_c = f$ *and any* $c' \in C^+_{n2}(f)$ *holds*

$$c_\emptyset = c'_{\emptyset\emptyset} + \sum_{j=1}^{n} c'_{\emptyset\{j\}} + \sum_{\{j,k\} \in \binom{\{1,\ldots,n\}}{2}} c'_{\emptyset\{j,k\}}$$

$$\forall j \in \{1,\ldots,n\} \quad c_{\{j\}} = c'_{\{j\}\emptyset} - c'_{\emptyset\{j\}} + \sum_{k \in \{1,\ldots,n\} - \{j\}} \left( c'_{\{j\}\{k\}} - c'_{\emptyset\{j,k\}} \right)$$

$$\forall \{j,k\} \in \binom{\{1,\ldots,n\}}{2} \quad c_{\{j,k\}} = c'_{\{j,k\}\emptyset} + c'_{\emptyset\{j,k\}} - c'_{\{j\}\{k\}} - c'_{\{k\}\{j\}}$$

**Proof.**

- Expansion of the posiform $c'$ yields a quadratic multi-linear polynomial form.
- Comparison with $c$ yields the conditions stated in the Lemma.

## Definition 10 (Complementation)

*For any* $n \in \mathbb{N}$ *and any QPBF* $f : \{0,1\}^n \to \mathbb{R}$,

$$r_f := \max_{c' \in C^+_{n2}(f)} c'_{\emptyset\emptyset} \tag{17}$$

*is called the* **floor dual** *of* $f$.

Lemma 7

*For any $n \in \mathbb{N}$ and any QPBF $f : \{0,1\}^n \to \mathbb{R}$, the floor dual can be computed in polynomial time.*

**Proof.** For the multi-linear polynomial form $c \in C_{n2}$ such that $f_c = f$, $r_f$ is the solution of the linear programming problem below (by Lemma 6).

$$
\begin{aligned}
\max_{c' : J_{n2}^{+} \to \mathbb{R}} \quad & c_{\emptyset} - \sum_{j=1}^{n} c'_{\emptyset\{j\}} - \sum_{\{j,k\} \in \binom{\{1,\ldots,n\}}{2}} c'_{\emptyset\{j,k\}} \\
\text{subject to} \quad & \forall j \in \{1,\ldots,n\} \quad c_{\{j\}} = c'_{\{j\}\emptyset} - c'_{\emptyset\{j\}} + \sum_{k \in \{1,\ldots,n\} - \{j\}} \left( c'_{\{j\}\{k\}} - c'_{\emptyset\{j,k\}} \right) \\
& \forall \{j,k\} \in \binom{\{1,\ldots,n\}}{2} \quad c_{\{j,k\}} = c'_{\{j,k\}\emptyset} + c'_{\emptyset\{j,k\}} - c'_{\{j\}\{k\}} - c'_{\{k\}\{j\}} \\
& \forall J \in J_{n2}^{+} - \{(\emptyset,\emptyset)\} \quad 0 \le c'_J \ .
\end{aligned}
$$

Can the floor dual be computed more efficiently than by an algorithm for general LPs?

## Definition 11

*For any* $n \in \mathbb{N}$ *and any* $c \in C^{+}_{n2}$, *the* **network** $N = (V, E, s, t, w)$ **of** $c$ *contains the nodes* $V = \{s, t, 1, \bar{1}, \dots, n, \bar{n}\}$ *and the weighted edges*

| | | |
|---|---|---|
| *for any* $c_{\{j\}\emptyset} > 0$ | $s\bar{j}, jt$ | $w_{s\bar{j}} := w_{jt} := \frac{1}{2} c_{\{j\}\emptyset}$ |
| *for any* $c_{\emptyset\{j\}} > 0$ | $sj, \bar{j}t$ | $w_{sj} := w_{\bar{j}t} := \frac{1}{2} c_{\emptyset\{j\}}$ |
| *for any* $c_{\{j,k\}\emptyset} > 0$ | $j\bar{k}, k\bar{j}$ | $w_{j\bar{k}} := w_{k\bar{j}} := \frac{1}{2} c_{\{j,k\}\emptyset}$ |
| *for any* $c_{\{j\}\{k\}} > 0$ | $jk, \bar{k}\bar{j}$ | $w_{jk} := w_{\bar{k}\bar{j}} := \frac{1}{2} c_{\{j\}\{k\}}$ |
| *for any* $c_{\emptyset\{j,k\}} > 0$ | $\bar{j}k, \bar{k}j$ | $w_{\bar{j}k} := w_{\bar{k}j} := \frac{1}{2} c_{\emptyset\{j,k\}}$ |